Equidistribution of periodic points for modular correspondences

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Abstract

Let T be an exterior modular correspondence on an irreducible locally symmetric space X. In this note, we show that the isolated fixed points of the power T^n are equidistributed with respect to the invariant measure on X as n tends to infinity. A similar statement is given for general sequences of modular correspondences.

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1 Introduction

intro

Let G be a connected Lie group and $\Gamma \subset G$ be a torsion-free lattice. Let $\widehat{\lambda}$ denote the probability measure on $\widehat{X} := \Gamma\backslash G$ induced by the invariant measure on G. Consider also an element $g \in G$ such that $g^{-1}\Gamma g$ is commensurable with Γ, that is, $\Gamma_g := g^{-1}\Gamma g \cap \Gamma$ has finite index in Γ. Denote by d_g this index.

The map $x \mapsto (x, gx)$ induces a map from $\Gamma_g\backslash G$ to $\widehat{X} \times \widehat{X}$. Let $\widehat{Y}_g$ be its image. The natural projections $\widehat{\pi}_1, \widehat{\pi}_2$ from $\widehat{Y}_g$ onto the factors of $\widehat{X} \times \widehat{X}$ define two coverings of degree d_g. Both of them are Riemannian with respect to every left-invariant Riemannian metric on G. The correspondence $\widehat{T}_g$ on $\widehat{X}$ associated with $\widehat{Y}_g$ is called *irreducible modular*.

A general modular correspondence $\widehat{T}$ on $\widehat{X}$ is a finite sum of irreducible ones, i.e. $\widehat{T}$ is associated with a sum $\widehat{Y} = \widehat{Y}_{g_1} + \cdots + \widehat{Y}_{g_m}$ that we call *the graph* of $\widehat{T}$. The degree d of $\widehat{T}$ is the sum of the degrees of $\widehat{T}_{g_i}$. We refer the reader to ClozelUtal, ClozelUllmo, Margulis [4, 5, 10] for more details.

If a is a point in $\widehat{X}$, define $\widehat{T}(a) := \widehat{\pi}_2(\widehat{\pi}_1^{-1}(a))$ and $\widehat{T}^{-1}(a) := \widehat{\pi}_1(\widehat{\pi}_2^{-1}(a))$. They are sums of d points which are not necessarily distinct. If $\widehat{U}$ is a small neighbourhood of a, the restriction of $\widehat{T}$ to $\widehat{U}$ can be identified to d local isometries $\widehat{\tau}_i : \widehat{U} \to \widehat{U}_i$ from $\widehat{U}$ to neighbourhoods $\widehat{U}_i$ of points a_i in $\widehat{T}(a)$. All these isometries

are induced by left-multiplication by elements of G. If a is a fixed point of $\widehat{\tau}_i$, i.e. $a = a_i$, we say that a is *a fixed point* of $\widehat{T}$. When a is an isolated fixed point of τ_i we also say a is *an isolated fixed point* of T. These points are repeated according to their multiplicities.

The composition $\widehat{T} \circ \widehat{S}$ of two modular correspondences $\widehat{T}$ and $\widehat{S}$ can be obtained by composing the above local isometries. This is also a modular correspondence. Its degree is equal to $\deg(\widehat{T})\deg(\widehat{S})$. Even when $\widehat{T}$ and $\widehat{S}$ are irreducible, their composition is not always irreducible. Denote by $\widehat{T}^n := \widehat{T} \circ \cdots \circ \widehat{T}$, n times, *the iterate of order* n of $\widehat{T}$. *Periodic points of order* n of $\widehat{T}$ are fixed points of $\widehat{T}^n$.

Let μ be a probability measure on $\widehat{X}$. Define a positive measure $\widehat{T}_*(\mu)$ of mass d on $\widehat{X}$ by

$$\widehat{T}_*(\mu) := (\widehat{\pi}_2)_*(\widehat{\pi}_1)^*(\mu).$$

A sequence of correspondences $\widehat{T}_n$ of degree d_n is said to be *equidistributed* if for any $a \in \widehat{X}$ the sequence of probability measures $d_n^{-1}(\widehat{T}_n)_*(\delta_a)$ converges weakly to $\widehat{\lambda}$ as n tends to infinity. Here, δ_a denotes the Dirac mass at a.

Let K be a compact Lie subgroup of G. Since the left-multiplication on G commutes with the right-multiplication, a modular correspondence $\widehat{T}$ as above, induces a modular correspondence T on $X := \widehat{X}/K$ with the same degree. Its graph is the projection Y of $\widehat{Y}$ on $X \times X$. The above notion and description of $\widehat{T}$ can be extended to T without difficulty. We call $\widehat{T}$ *the lift* of T to $\widehat{X}$. Consider on X the probability measure λ induced by the invariant measure on G, i.e. the direct image of $\widehat{\lambda}$ in X. Here is our main result.

th_main **Theorem 1.1.** *Let T_n be a sequence of modular correspondences on X and let $\widehat{T}_n$ be the lifts of T_n to $\widehat{X}$. Assume that the sequence $\widehat{T}_n$ is equidistributed. Then the isolated fixed points of T_n are equidistributed. More precisely, there is a constant $s \geq 0$, depending only on G and K, such that if d_n is the degree of T_n and P_n is the set of isolated fixed points of T_n counted with multiplicity, we have*

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{a\in P_n} \delta_a = s\lambda.$$

The last convergence is equivalent to the following property. If W is an open subset of X such that its boundary has zero λ measure, then

$$\lim_{n\to\infty} \frac{|P_n \cap W|}{d_n} = s\lambda(W).$$

We can of course replace W with $\overline{W}$.

Now, assume moreover that G is semi-simple, K is a maximal compact Lie subgroup of G and Γ is an irreducible lattice. An irreducible correspondence T associated with an element $g \in G$ as above is *exterior* if the group generated by g and Γ is dense in G. For such a correspondence, Clozel-Otal proved in [4] that the iterate sequence $\widehat{T}^n$ is equidistributed (their proof given for T is also valid for $\widehat{T}$), see also Clozel-Ullmo [5]. We deduce from Theorem 1.1 the following result.

Corollary 1.2. *Let T be an exterior correspondence on an irreducible locally symmetric space X as above. Then the isolated periodic points of order n of T are equidistributed with respect to λ as n tends to infinity.*

The proof of our main result will be given in Section 2. In Section 3, we will give similar results related to the Arnold-Krylov-Guivarc'h theorem [1, 8]. We refer to Benoist-Oh [2] and Clozel-Oh-Ullmo [3] for other sequences of modular correspondences for which our main result can be applied. The reader will also find in Clozel-Ullmo [5], Dinh-Sibony [6, 7] and Mok-Ng [11, 12] some related topics.

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2 Proof of the main result

Fix a Riemannian metric on G which is invariant under the left-action of G and the right-action of K. It induces Riemannian metrics on $\widehat{X}$ and X. We normalize the metric so that the associated volume form on X is a probability measure. So, it is equal to λ. If $\Pi : \widehat{X} \to X$ is the canonical projection, we have $\Pi_*(\widehat{\lambda}) = \lambda$. Let l and m denote the dimension of G and X respectively.

Fix a point $c \in X$. Denote by $B(c, r)$ the ball of center c and of radius r in X. In order to prove the main result, we will consider the following quantity

$$\frac{|P_n \cap B(c, r)|}{d_n}.$$

Let Φ denote the natural projection from G to $\widetilde{X} := G/K$. The image of K by Φ is a point that we denote by 0. Denote by $B(0, r)$ the ball of center 0 and of radius r in $\widetilde{X}$. Define $K_r := \Phi^{-1}(B(0, r))$. So, K_r is a union of classes xK with $x \in G$. Fix also a constant $r_0 > 0$ small enough so that $B(0, r')$ is convex for every $r' \leq 3r_0$. Here, the convexity is with respect to the Riemannian metric induced by the one on G. From now on, assume that $r < r_0$.

Lemma 2.1. *Let g be an element of G. If g admits a fixed point in $B(0, r)$ then g belongs to K_{2r}. The set of fixed points of g in $B(0, r_0)$ is a convex submanifold of $B(0, r_0)$. Moreover, a fixed point $e \in B(0, r_0)$ of g is isolated if and only if 1 is not an eigenvalue of the differential of g at e.*

Proof. Assume that g admits a fixed point e in $B(0, r)$. Since g is locally isometric, $g(0)$ belongs to $B(0, 2r)$. It follows that g belongs to K_{2r}. If e, e' are two different fixed points in $B(0, r_0)$ then every point of the geodesic in $B(0, r_0)$ containing e, e' is fixed. We deduce that the set of fixed points in $B(0, r_0)$ is a

convex submanifold. If 1 is an eigenvalue of the differential of g at e, the associated tangent vector at e defines a geodesic of fixed points. This implies the last assertion in the lemma. $\square$

Recall that *a semi-analytic set* in a real analytic manifold W is locally defined by a finite family of inequalities $f > 0$ or $f \geq 0$ with f real analytic. A set in W is *subanalytic* if locally it is the projection on W of a bounded semi-analytic set in $W \times \mathbb{R}^n$. The boundary of a subanalytic open set is also subanalytic with smaller dimension. We refer the reader to [9] for further details. We will need the following lemma.

a_subanalytic **Lemma 2.2.** *Let M_r denote the set of all $g \in G$ which admit exactly one fixed point in $B(0,r)$. Then M_r is a subanalytic open set contained in K_{2r}.*

Proof. The last assertion in Lemma 2.1 implies that M_r is open. The first assertion of this lemma implies that M_r is contained in K_{2r}.

Denote by M' the set of points (g,x) in $K_{2r_0} \times B(0,r_0)$ such that $g(x) = x$. This is an analytic subset of $K_{2r_0} \times B(0,r_0)$. So, it is a semi-analytic set in $G \times \widetilde{X}$. Let M be the set of points (g,x) in M' such that the differential of g at x does not have 1 as eigenvalue. So, M is also a semi-analytic set.

If σ_1, σ_2 are the natural projections from M' to G and to $\widetilde{X}$ respectively, we deduce from Lemma 2.1 that M_r is equal to $\sigma_1(M \cap \sigma_2^{-1}(B(0,r)))$. Moreover, σ_1 defines a bijection from $M \cap \sigma_2^{-1}(B(0,r))$ to M_r. It is now clear that M_r is a subanalytic set. $\square$

Consider a general modular correspondence T as above. Let π_1, π_2 denote the natural projections from Y to X. If r is small enough, the ball $B(c,r)$ is simply connected and $\pi_1^{-1}(B(c,r))$ is the union of d balls $B(c_i', r)$ of center c_i' in Y. The restriction of π_1 to $B(c_i', r)$ is injective. The projection π_2 sends $B(c_i', r)$ to the ball $B(c_i, r)$ of center $c_i := \pi_2(c_i')$ in X. So, the restriction of T to $B(c,r)$ is identified with the family of d maps $\tau_i : B(c,r) \to B(c_i, r)$.

Fix a point $b \in \widehat{X}$ such that $\Pi(b) = c$. Let $\widehat{T}$ denote the lift of T to $\widehat{X}$ as above. The restriction of $\widehat{T}$ to $B(b,r)$ can be identified with a family of d maps $\widehat{\tau}_i : B(b,r) \to B(b_i, r)$ which are the lifts of τ_i to $\widehat{X}$, i.e. we have $\Pi \circ \widehat{\tau}_i = \tau_i \circ \Pi$.

Fix also a point $a \in G$ such that $\Psi(a) = b$ where $\Psi : G \to \widehat{X}$ is the natural projection. The left-multiplication by a induces the map $x \mapsto \Psi(ax)$ from M_r to $\widehat{X}$. Its image is independent of the choice of a and is denoted by $M_{b,r}$. Since Γ is torsion-free, its intersection with K is trivial. Therefore, when r is small enough, the above map is injective on K_{2r}. So, it defines a bijection from M_r to $M_{b,r}$. This is an isometry since the metric on G is invariant.

lemma_fix_bis **Lemma 2.3.** *The map τ_i admits exactly one fixed point in $B(c,r)$ if and only if b_i belongs to $M_{b,r}$.*

Proof. Without loss of generality, we can assume that T and $\widehat{T}$ are irreducible and given by an element $g \in G$ such that $g^{-1}\Gamma g$ is commensurable with Γ. Choose d elements $\delta_1, \ldots, \delta_d$ of Γ which represent the classes of $\Gamma_g \backslash \Gamma$. Then, up to a permutation, $\widehat{\tau}_i$ and τ_i are induced by the maps $x \mapsto g_i x$ where $g_i := g\delta_i$.

Assume that τ_i has a unique fixed point in $B(c, r)$. This point can be written as $\Theta(ae)$ for some point $e \in B(0, r)$, where Θ is the canonical projection from $\widetilde{X}$ to X. So, we have $g_i ae = \gamma ae$ for some $\gamma \in \Gamma$. The maps $\widehat{\tau}_i$ and $\widehat{\tau}$ are also induced by $x \mapsto g_i' x$ where $g_i' := \gamma^{-1} g_i$ since $\gamma^{-1} \in \Gamma$. We have $g_i' ae = ae$ and $(a^{-1} g_i' a)e = e$. By Lemma 2.1, $a^{-1} g_i' a$ belongs to M_r. Since $b_i = \Psi(g_i' a)$, we deduce that $b_i \in \Psi(aM_r) = M_{b,r}$. We see in the above arguments that the converse is also true. $\square$

End of the proof of Theorem 1.1. Denote by λ' the volume form on G which induces on $\widehat{X}$ the form $\widehat{\lambda}$. By Lemma 2.2, M_r and $M_{b,r}$ are subanalytic sets. So, their boundaries are of dimension $\leq l - 1$. Since the sequence $\widehat{T}_n$ is equidistributed, using Lemma 2.3, we obtain

$$\lim_{n\to\infty} \frac{|P_n \cap B(c, r)|}{d_n} = \lim_{n\to\infty} \frac{|\widehat{T}_n(b) \cap M_{b,r}|}{d_n} = \widehat{\lambda}(M_{b,r}) = \lambda'(M_r).$$

It follows that the sequence of positive measures

$$\frac{1}{d_n} \sum_{x \in P_n} \delta_x$$

converges to a measure μ which satisfies $\mu(B(c, r)) = \lambda'(M_r)$ for r small enough. Since M_r is contained in K_{2r}, the last quantity is of order $O(r^m)$. Hence, $\mu = s\lambda$ where $s \geq 0$ is a function. Finally, the fact that $\lambda'(M_r)$ is independent of c implies that s is constant. It depends only on G and K. $\square$

Remark 2.4. The constant s is an invariant depending only on G and K. So, it can be computed using a particular case, e.g. when Γ is co-compact and T_n have only isolated fixed points. So, Lefschetz's fixed points formula may be used here. We have for example $s = 2$ when $G = \mathrm{PSL}(2, \mathbb{R})$ and $K = \mathrm{SO}(2)$. We can also obtain a speed of convergence in our main theorem in term of the speed of convergence in the equidistribution property of $\widehat{T}_n$.

3 On the Arnold-Krylov-Guivarc'h theorem

Consider now the case where G is a compact connected semi-simple Lie group, Γ is trivial and K a connected compact subgroup of G. Define $X := G/K$. Let $\widehat{\lambda}$ be the invariant probability measure of G and λ its direct image in X.

Let $H \subset G$ be a semi-group generated by a finite family of elements $g_1, \ldots, g_d$ of G. Denote by H_n the set of words of length n in H. We say that H is *equidistributed* on G if for every point $a \in G$, the sequence of probability measures

$$d^{-n} \sum_{g \in H_n} \delta_{ga}$$

converges to $\widehat{\lambda}$ as n tends to infinity.

The left-multiplication by g_i defines a self-map $\widehat{T}_{g_i}$ on G. Their sum $\widehat{T}$ can be seen as a correspondence of degree d on G. It induces a correspondence T on X of the same degree. So, H is equidistributed if and only if the sequence $\widehat{T}^n$ is equidistributed. We deduce from our main result the following theorem.

Theorem 3.1. *Let G, K, X, λ, H and H_n be as above. Assume that H is equidistributed on G. Then the isolated fixed points in X of the elements of H_n are equidistributed with respect to λ when n tends to infinity.*

Assume that $d = 2$ and that the first Betti number of X vanishes. A result by Guivarc'h [8] says that if the group generated by H is dense in G then H is equidistributed, see also Arnold-Krylov [1]. So, Theorem 3.1 can be applied in this case.

A similar result holds for groups. Let $H \subset G$ be a group generated by a finite family $\{g_1, \ldots, g_{2d}\}$ where $g_i = g_{2d-i}^{-1}$. Let H_n denote the family of reduced words of length n in H. We say that H is *equidistributed* if the sequence of probability measures

$$\mu_n := \frac{1}{|H_n|} \sum_{g \in H_n} \delta_{ga}$$

converges to $\widehat{\lambda}$ for every $a \in G$. There are also correspondences $\widehat{T}_n$ and T_n of degree $|H_n|$ such that $(\widehat{T}_n)_*(\delta_a) = |H_n|\mu_n$. So, Theorem 3.1 holds for equidistributed groups H.

Another result by Guivarc'h [8] says that if $d = 2$ and if H is dense in G then it is equidistributed. Therefore, our result can be applied under these conditions.

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